

A suggestion for an integrability notion for two dimensional spin systems

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Abstract

We suggest that trialgebraic symmetries might be a sensible starting point for a notion of integrability for two dimensional spin systems. For a simple trialgebraic symmetry we give an explicit condition in terms of matrices which a Hamiltonian realizing such a symmetry has to satisfy and give an example of such a Hamiltonian which realizes a trialgebra recently given by the authors in another paper. Besides this, we also show that the same trialgebra can be realized on a kind of Fock space of q -oscillators, i.e. the suggested integrability concept gets via this symmetry a close connection to a kind of noncommutative quantum field theory, paralleling the relation between the integrability of spin chains and two dimensional conformal field theory

1 Introduction

We should stress immediately at the beginning that what we discuss on the following pages is a proposal for an integrability notion for two dimensional rectangles of spins which we believe contains necessary features such a notion should have and has, in addition, the merit of a certain naturality. We do not claim at the present stage that this is the final word in the important question of integrability notions for higher dimensional spin systems than spin chains. It is an open question if there exists any sensible integrability notion in the classical sense of conserved quantities for such systems. What we show is that there exists a concept - which one might maybe call “weak integrability” - which is linked to a symmetry structure which relates it to four dimensional topological field theory and q -deformed quantum field theory, much the same way the quantum group symmetries of integrable spin chain models relate these to three dimensional topological field theory and two dimensional conformal field theory.

2 Spin rectangles

Trialgebras are algebraic structures with two associative products and a coassociative coproduct, all linked in a compatible way. Trialgebras relate to bialgebras as do Hopf algebras to groups. The explicit examples of trialgebras constructed in [GS] and [GS2] can be seen as a kind of second quantization of quantum groups. Concretely:

Definition 1 *A trialgebra $(A, *, \Delta, \cdot)$ with $*$ and \cdot associative products on A (where $*$ may be partially defined, only) and Δ a coassociative coproduct on A is given if both $(A, *, \Delta)$ and (A, \cdot, Δ) are bialgebras and the following compatibility condition between the products is satisfied for arbitrary elements $a, b, c, d \in A$:*

$$(a * b) \cdot (c * d) = (a \cdot c) * (b \cdot d)$$

whenever both sides are defined.

An example where $*$ is partially defined is given by starting from a bialgebra and passing to its tensor algebra. i.e. taking the tensor product as the

second product. In physics parlance the partial definition of $*$ means that we can only multiply “ n particle” with “ m particle” states for $n = m$ but not for $n \neq m$. We will only encounter trialgebras which are partially defined in precisely this way, in the sequel. In [GS] we gave a formulation of the above compatibility condition for \cdot and $*$ in terms of R -matrices R_p and R_q if both bialgebras, $(A, *, \Delta)$ and (A, \cdot, Δ) , are quantum groups with R -matrices R_p and R_q , respectively. In analogy to quantum groups we will call trialgebras of this kind *second quantized quantum groups*.

It is generally be viewed an essential feature of integrability for a one dimensional spin chain if the algebraic structure (Hamiltonian and symmetries) can systematically be derived for a chain with $n + 1$ spins from one with n spins which is exactly satisfied in the presence of a quantum groups symmetry (as e.g. in the XXZ model). In this case it is the coproduct which allows to the data for the longer chain.

Integrability for a two dimensional spin system should then naturally mean that we can systematically derive the data of larger and larger rectangles of spins from smaller ones. But this means, especially, that the dimensional reduction on the two possible directions has to lead to integrable spin chain models. So, we have two quantum groups for the two integrable spin chains. In addition, we have to have the above compatibility of the two products because when enlarging both edges of a spin rectangle the result should not depend upon the choice which edge we start to enlarge first, i.e. the two products have to interchange. Since if we enlarge only one edge, we enlarge already the whole rectangle, the coproduct necessarily has to be one and the same for both quantum groups. In conclusion, we should get a second quantized quantum group whenever we have an integrable spin rectangle.

Remark 1 *Observe that also the structure of a partially defined $*$ product, of the kind discussed above, has a natural interpretation in this context: It simply means that we have a rectangle of spins and not some other two dimensional array of spins.*

If one accepts a second quantized quantum group as a suggestion for an integrability criterion for spin rectangles, the next question is how one could construct a Hamiltonian for a spin rectangle which embodies such a symmetry. As a simple example, we take the trialgebraic deformation

$\check{U}_{p,q}^F(sl_2)$ of the quantum group $\check{U}_q(sl_2)$ constructed in [GS] (for the definitions and details, see there). The requirement that dimensional reduction on the two possible directions has to lead to integrable spin models, leads to a decisive restriction for the Hamiltonian, too. The Hamiltonian of the XXZ-model is given by

$$H_{XXZ} = \sum_i (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+) + \lambda_q \sum_i (\sigma_i^z \sigma_{i+1}^z - 2) + \sum_i (\sigma_i^z - \sigma_{i+1}^z)$$

where λ_q is a constant involving the deformation parameter q , the index is running over the n elements of the chain and e.g. σ_i^+ stands for

$$\sigma_i^+ = \mathbf{1} \otimes \dots \otimes \mathbf{1} \otimes \sigma^+ \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1}$$

with σ^+ occuring in the i -th place. If we introduce a second label j for the second direction in the spin rectangle, the complete Hamiltonian H should have the form

$$H = \sum_{i,j} (\sigma_{i,j}^+ \sigma_{i+1,j}^- + \sigma_{i,j}^- \sigma_{i+1,j}^+) + \lambda_q \sum_{i,j} (\sigma_{i,j}^z \sigma_{i+1,j}^z - 2) + \sum_{i,j} (\sigma_{i,j}^z - \sigma_{i+1,j}^z) + \sum_i H_{p,i}$$

with the obvious meaning of e.g. $\sigma_{i,j}^+$ and H_p the Hamiltonian for the spin chain in the second direction, involving the deformation parameter p . Dimensional reduction on the j -direction, i.e. taking an inner trace over i (where by an inner trace we mean that in the i direction tensor products are converted into matrix products), leads - modulo a constant factor given by the number of spins in the i -direction and a constant shift - to

$$\begin{aligned} & \sum_j (\sigma_j^+ \sigma_j^- + \sigma_j^- \sigma_j^+) + \lambda_q \sum_j \sigma_j^z \sigma_j^z + H_p \\ &= \sum_j (\sigma^+ \sigma^- + \sigma^- \sigma^+)_j + \lambda_q \sum_j (\sigma^z)_j^2 + H_p \\ &= m(1 + \lambda_q) \mathbf{1} + H_p \end{aligned}$$

which as a Hamiltonian is equivalent to H_p . So, we get the correct dimensional reduction, here. Dimensional reduction on the i -direction - taking an inner trace over j - also reproduces H_{XXZ} correctly, provided the inner trace over j of H_p is a multiple of $\mathbf{1}$. In order to realize the trialgebraic symmetry, H_p in addition has to commute with a representation of the associative

algebra structure with respect to the \cdot product of $\check{U}_{p,q}^F(sl_2)$. In conclusion, any matrix H_p realizing this commutator constraint together with the above inner trace condition, would lead to realization of the trialgebraic symmetry on a spin rectangle system and would therefore lead to a system which naturally suggests itself as an integrable two dimensional spin model.

Remark 2 *In the case of integrability for spin chains - where the usual quantum group symmetries arise - there is an important connection to three dimensional topological quantum field theories which are determined by just the same quantum groups. This close connection to a topological theory can be seen as a clear sign of integrability. It is therefore interesting that our proposal for an integrability notion for spin rectangles, based on trialgebraic symmetries, leads to a close connection to the algebraic structure of four dimensional quantum field theories. Trialgebras lead to bialgebra categories as their categories of representations and with certain types of trialgebras (the one used below being an example) lead to so called Hopf algebra categories. But Hopf algebra categories have been shown to be suitable to generate four dimensional topological quantum field theories (for the details of the notions and claims of this remark, see [CrFr], [CKS], [GS]).*

As a concrete example for H_p , start from the algebra with respect to the p -deformed \cdot product given in [GS] by the table

$$\begin{array}{rcl} a \cdot b & = & p^{-1} \quad b \cdot a \\ a \cdot c & = & p \quad c \cdot a \\ a \cdot d & = & d \cdot a \\ b \cdot c & = & p^2 \quad c \cdot b \\ b \cdot d & = & p \quad d \cdot b \\ c \cdot d & = & p^{-1} \quad d \cdot c \end{array}$$

(for the generators of $U_q(sl_2)$, we get a completely similar table, so, without loss of generality we can - as concerns the p -deformed product - work with the a, b, c, d , only). Observe that one can get a representation of the a, b, c, d in the following way: Let $\{|n\rangle, n \in \mathbb{N}\}$ dentote a basis and Λ the shift operator, i.e.

$$\Lambda |n\rangle = |n+1\rangle$$

Let

$$a = d$$

with

$$a |n\rangle = p^{n-1} |n\rangle$$

and

$$\begin{aligned} b &= \Lambda^{-1}a \\ c &= a\Lambda \end{aligned}$$

This defines a representation for the above table. For p a root of unity this generates a finite dimensional representation. The R matrix corresponding to the above table is

$$R_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}$$

In analogy to the usual quantum group case, we take

$$H_p = \sum_j \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}_j \otimes \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}_{j+1}$$

then. Here, we assume p to be a root of unity and the length of the chain to be adjusted in such a way that we have the above finite dimensional representation on the complete state space. Assuming periodic boundary conditions for the j direction, H_p commutes with the a, b, c, d . Besides this, the inner contraction of H_p over j is a multiple of

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and all the requirements on H_p are satisfied, therefore. Hence, we have found a model of a spin plane which realizes one of the trialgebraic symmetries constructed in [GS].

3 q -oscillators

We are now going to show that one can also realize the trialgebra $\check{U}_{p,q}^F(sl_2)$ of [GS] on a deformation of the q -oscillator algebra. We take the following

version of the q -oscillator algebra: The algebra \mathcal{A}_q is the complex associative unital algebra with generators a, a^+, q^N, q^{-N} and relations

$$\begin{aligned} q^N q^{-N} &= q^{-N} q^N = 1 \\ q^N a &= q^{-1} a q^N \\ q^N a^+ &= q a^+ q^N \end{aligned}$$

and

$$\begin{aligned} [a, a^+]_q &\equiv aa^+ - qa^+a = q^{-N} \\ [a, a^+]_{q^{-1}} &\equiv aa^+ - q^{-1}a^+a = q^N \end{aligned}$$

(see e.g. [KS]). Let T be an element of the continuous family of algebra morphisms which exist from the quantum algebra $\check{U}_q(sl_2)$ into the q -oscillator algebra \mathcal{A}_q (see [KS]). More concretely, this family is given by (for $\alpha \in \mathbb{C}$)

$$\begin{aligned} T_\alpha(E) &= a \\ T_\alpha(F) &= a^+ [N - 2\alpha]_q \\ T_\alpha(K) &= q^{N-\alpha} \end{aligned}$$

where $[x]_q$ denotes the usual q -number expression for x .

Lemma 1 *The deformation $\check{U}_{p,q}^F(sl_2)$ of $\check{U}_q(sl_2)$ defines a unique Fock space deformation $\mathcal{A}_{p,q}^F$ of \mathcal{A}_q such that T extends also to a morphism of the \cdot product and the two products of $\mathcal{A}_{p,q}^F$ (the one from \mathcal{A}_q and the deformation of the symmetric tensor product) are compatible.*

Proof. By the defining relations of $\check{U}_{p,q}^F(sl_2)$, we get only a noncommutative behaviour for the exchange of a and a^+ and for the exchange of one of these elements with the unit (which is not a unit for the \cdot product but only for the $*$ product, as we remarked in [GS], [GS2]). Precisely, we get

$$\begin{aligned} a \cdot 1 &= p \, 1 \cdot a \\ 1 \cdot a^+ &= p \, a^+ \cdot 1 \\ a \cdot a^+ &= p^2 \, a^+ \cdot a \end{aligned}$$

The compatibility of the two products follows by calculation from this. ■

Observe that - analogous to the fact that \mathcal{A}_q is not a quantum group - $\mathcal{A}_{p,q}^F$ is not a trialgebra but an algebraic structure with two compatible products, only. One can imagine $\mathcal{A}_{p,q}^F$ as a collection (infinite, if p is not a root of unity) of q -oscillators where the p -deformation of the tensor product has introduced a certain kind of interaction between them. In this sense it is a kind of very simple toy model version of a noncommutative quantum field theory and shows that trialgebraic symmetries do, indeed, appear in this context. For the simple trialgebraic deformations with a scalar deformation function constructed in [GS] and [GS2], we can certainly not expect more than a relation to such toy models. More physically realistic theories will need more complicated trialgebras to describe their symmetries.

Let us spent a few more remarks on the “interaction” introduced by the statistics of this deformed tensor product (i.e. the \cdot product): We have

$$a \cdot 1 \cdot a^+ = p^4 a^+ \cdot 1 \cdot a$$

and more generally

$$a \cdot 1 \cdot \dots \cdot 1 \cdot a^+ = p^{2(l-1)} a^+ \cdot 1 \cdot \dots \cdot 1 \cdot a$$

where the number of units in the middle position is l . So, for $0 < p < 1$ we have an “interaction” which is decreasing with increasing separation of the q -oscillators (if we interpret the different positions in formal words with respect to the \cdot product as a kind of distance).

If we introduce for an n factor expression the notation

$$a_{i,n} = 1 \cdot \dots \cdot 1 \cdot a \cdot 1 \cdot \dots \cdot 1$$

if a appears in the i -the position, we get - by compatibility of the two products - the following behaviour under the $*$ product:

$$a_{i,n} * a_{j,n}^+ = a_{j,n}^+ * a_{i,n}$$

and, in consequence, the rule

$$a_{i,n} * a_{j,n}^+ - q a_{j,n}^+ * a_{i,n} = q^{-N} \delta_{ij}$$

for the $*$ product commutator of different q -oscillators.

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